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SPHERICAL GEOMETRY.

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LECTURE VII. CIRCLES, ROTATIONS, AND TANGENTS.

The properties of circles on the sphere are in a very large measure analogous to the properties of circles in the plane. There are, however, a number of minor differences which we shall touch upon; there are some things that are new; and there are a considerable number of theorems and points of view which are not taken up in the ordinary treatment. It is especially on account of these last that we take up the subject.

The ordinary definition of a circle as the locus of points at a given distance from a fixed point called the center would apply equally well to the sphere if we had developed the theory of distance.* But as we are not yet in possession of any measure of distance the definition may better be stated in another form as: The circle is the locus of all possible positions of one end of a given segment when the other end remains fixed at a point called the center of the circle. Or, if the idea of rotation is brought in, *the circle is the locus described by one end of a segment which is rotated about the other end as center.* The segment in any of its positions is called the radius of the circle. It may be noted that the point antipodal to the center also possesses all the properties of the center, and

*Throughout the following pages we speak of the sum and difference of segments. These words are used in a purely geometrical sense and their use may be justified in a manner such as that given on page 127. To think of the sum of two segments as a segment whose length is the sum of the lengths of the two segments is not necessary. It is not even advisable and until we have developed the theory of measure it is impossible. For this reason we use the term *congruent* in place of *equal*.

that the radii drawn from the two centers, when taken together, fill out a semi-line. If in particular the radius of a circle is a quadrant, the circle becomes a great circle, that is, a line. As a matter of convenience we shall agree that when we speak of the center and the radius of a circle, not a great circle, we shall mean *that* radius which is less than a quadrant and the center to match. It will be left to the reader to convince himself in how far our work applies also to the other of the two cases.

Theorem 61. *If either of the two perpendiculars, dropped from the center of a circle, to a given line is less than the radius of the circle, the line has two points in common with the circle; if either is congruent to the radius, the line has one point in common with the circle; if both are greater than the radius, the line has no point in common with the circle.*

The proof of this theorem depends on Theorem 60 and the principle of continuity. Let C be the center of the circle, A and B the points where the perpendiculars from C meet the given line. Consider one of the semi-lines AB. Let CA be the perpendicular which is less than the radius. Then it is easy to show that CB is greater than the radius. Let P be any point of the semi-line in question. As P describes the line from A to B, the segment CP is constantly increasing (Theorəm 60). The points P may be divided into two classes, namely, those for which CP is less than or congruent to a radius and those for which it is greater than the radius. Every point of the first class precedes all points of the second and hence there exists a point such that every point which precedes it is of the first class and every point which succeeds it is of the second. This point is such that for it CP is congruent to the radius. In like manner there exists a similar point in the other segment AB, and the first part of the theorem is proved. The remaining parts are left to the reader.

Theorem 62.* *If C be the center of a circle and r the radius, and if D and E are two points which satisfy the relation $CD<r<CE$, then the line DE cuts the circle in two points.*

To prove this theorem it is only necessary to drop the perpendiculars CA and CB on the line DE and note that CA is less than CD or congruent to it and that CB is greater than CE or congruent to it. Hence the case can be brought under the first case of Theorem 61. We see then that a circle divides the surface into two parts, an interior and an exterior, in the same sense as a triangle divides the surface (Cf. Lecture IV).

The large number of theorems concerning the perpendicular dropped from the center of a circle to a chord, the perpendicular erected at the middle point of a chord, the congruence of the perpendiculars from the center when the chords are congruent, and the relations of less than and greater than which occur when the chords are not congruent are all left to the reader, statement and proof.

Theorem 63. *The maximum and minimum segments which can be drawn from a point C' to a circle are the segments which connect C' to the points A and B in which*

*In some of these theorems and proofs it is not altogether possible to avoid a great similarity with the treatment of the same subject in the excellent "Questioni riguardanti elementare" edited by F. Enriques.

the circle is cut by the line CC' passing through the center and the given point; and as the point P moves along the circle from the point A to the point B, the segment CP steadily decreases.

The first part of this theorem is without difficulty.* The second part contains a number of points worthy of note and illustrating the care that must be exercised if one is not to fall into making gratuitous assumptions. In the first place there is the phrase "moves along the circle" which certainly implies that the points of a circle may be considered to be situated in a natural order. In fact we shall consider that the points of a circle are ordered—this order being taken directly from the order of the semi-lines or directions issuing from the center of the circle. As there is one and only one point on each semi-line and as the semi-lines are ordered this is both possidle and appropriate. In the second place the words "steadily decreases" are likely to imply "constantly and *continuously* decreasing." As a matter of fact nothing is said concerning the continuity. It is, however, easy to remedy the deficiency by making use of the continuity among the directions issuing from the center of the circle. We may state

Theorem 64. *If the points of a circle be divided into two classes such that every point of the first class precedes every point of the second class, then there exists one point such that any point which precedes it lies in the first class and every point which follows it lies in the second.*

From the intuitive point of view it seems evident that the points on a circle lie upon a smooth curve and are not scattered promiscously about from point to point on the surface. This, however, is something to be proved; and besides, we have not yet said what a "curve" is, nor what "smooth" means. Let us prove the following theorem.

Theorem 65. *On the circle there are no isolated points; that is, if there be drawn a circle of radius ever so small about any point of a given circle, then there are other points of the given circle within it.*

Let O be any point of the given circle. Describe about O as center a circle of arbitrary radius r. From M lay off a segment MN less than $\frac{1}{2}r$. Let C be the center of the given circle. Consider the triangle MCN. On the other side of CN construct the triangle $M'CN$ symmetrical to MCN. $M'C$ is congruent to MC and hence M' lies on the circle. The line MM' is perpendicular to CN and hence no greater than the sum of MN and $M'N$ which is congruent to it. Hence MM' is less than r; and the theorem is proved.

Theorem 66. *In the statement of Theorem 63 the words "steadily decreases" may be replaced by "steadily and continuously decreases"; that is: The segment CP takes on all possible values between the maximum CA and the minimum CB.*

The proof is given by dividing all the points of the semi-circle AB into two classes such that for every point P of the first class CP is greater than or congruent to any assigned segment a which is between CA and CB and for every

*The theorem that the sum of two sides of a triangle is greater than their third side and that the difference of two sides is less than the third side are used in the proof. They are such immediate corollaries of theorem 59, that we have not seen fit to mention them in the text.

point P of the second class CP is less than a. All points save the end-points A and B belong to one of the classes. All points of the first class precede the points of the second (Theorem 63). Therefore there exists a point P° as specified in Theorem 64. The segment CP° is congruent to the segment a. For suppose that CP° were greater than a. It is possible to find (Theorem 65) a point P' which follows P° and such that the chord $P^\circ P'$ is less than the difference between CP° and a. As the segment CP' is greater than the difference of CP° and and $P^\circ P'$, it is greater than a, although less than CP° on account of the fact that P' follows P°. Thus there is found a point P' which follows P°. Thus there is found a point P' which follows P° and is in the second class and which is also such that CP' is greater than a. The contradiction is apparent. In like manner CP° can be proved to be not less than a. It must therefore be congruent to a; and the theorem is proved.

From this theorem it follows as an immediate corollary that two circles intersect in two points if the sum of the radii is greater than the segment which joins the centers of the circles and if at the same time the difference of the radii is less than that segment. In case that the sum is congruent to the segment the circles have one point in common and all other points so situated that those on either circle lie without the other circle. In case the difference is congruent to that segment the circles likewise have one point in common, but the remaining points are so situated that all the points of one of the circles lie within the other. It may occur to some that these expressions are a long way of saying that the circles in the two cases are respectively internally and externally tangent. As yet, however, no mention has been made of tangents. The tangent is a more complicated conception, when analyzed carefully, than continuity and the relations of within and without, such as have been used up to this point.

Before leaving the class of questions with which we have been dealing it is well to note that the existence of the intersections of two circles which fulfill the stated conditions may serve to prove

Theorem 67. *Given three segments with the condition that the sum of two is greater than the third and the difference of those two less than the third, then there can be constructed two triangles (the one symmetrical to the other) having the three segments as sides.*

In most treatments the existence of such triangles is assumed instead of proved. We may also state that the theorem on the intersection of circles can be stated in the form: If two circles are so situated that there exists one point of one within and another point of the same one without the other, then the two circles have two points in common. The proof of this more general form is left to the reader. It establishes the fact that in the definition of "interior" in the case of a circle the two points may be joined by an arc of a circle, instead of by a line (see under Theorem 62 and in Lecture IV). We are also now in a position to give constructions for the middle point of a segment and for the bisector of an angle.

Adequately to discuss many of the more advanced parts of Spherical Ge-

ometry requires theorems analogous to those given in the previous Lecture (Theorems 55, 56) for segments. Namely, it is necessary to know the rules governing the addition and subtraction of angles. If AOB and $CO'D$ be two angles we shall understand by their sum the angle AOD which is obtained by bringing their vertices into coincidence in such a manner that the initial line OC of the second angle falls along the terminal line OB of the first angle. Evidently two cases arise according as the directions of rotation from OA to OB and from $O'C$ to $O'D$ are the same or different. These distinctions arise in the case of angles, although they are not present in the case of segments, owing to the difference between congruent and symmetrical angles. For the same reason the method of proof given for the addition of segments cannot be applied to the addition of angles. The proofs may, however, be obtained by using the results obtained for segments.

Theorem 68. *The sum $a+b$ of two angles is congruent to the sum $b+a$ of the same angles added in a different order.*

There are several cases which arise and which must be considered separately. These we leave to the reader with the hint that to effect a reduction to the theory of segments it is only necessary to describe a great circle about the vertex of the angle as center. Congruent or symmetrical angles intercept congruent segments in this great circle or line, and conversely congruent segments subtend congruent or symmetrical angles. The theorems concerning subtraction are obtainable directly from those concerning addition by replacing the angle to be subtracted by its symmetrical and adding. By means of this theorem it may be shown that when the surface is rotated about a point O, every direction issuing from O moves through the same angle. Thus an angle is sufficient to determino the amount of rotation about a point, without specifying any particular direction which is to be moved into some other specified direction. We might go on to show that any rigid motion of the surface is a rotation through a definite angle (or its symmetrical) and about a definite point (or its antipodal).

In discussing the subject of tangents in the special case of circles it is possible to give a special definition, namely, that: *A line drawn perpendicular to the radius at its extremity is tangent to the circle.* With such a definition the simpler geometric properties of tangents becomes evident: but it is by no means immediately demonstrable that the tangent is the limit of a secant when one of the two points of intersection of the secant and the circle approaches the other as a limit. It is not even evident that the limit exists. There are other problems in limits which occur naturally in elementary geometry. These are the determination of the length of a circle or the area of a circle. The length of a convex curve like the circle may be defined as the limit of the length of an inscribed or circumscribed polygon when the number of the sides increases indefinitely and the length of each side approaches zero as its limit. It is necessary to show that this limit exists and is independent of the way in which the polygon approaches the circle as its limit. The same is true in the case of area, which may be defined in a similar manner. As we have not yet established a measure of length

or area the discussion of the limits involved in their evaluation must be postponed. The treatment of all problems in limits belongs essentially to the differential and integral calculus; and it might be far better to leave these difficult questions until the analytic means for adequately handling them have been developed. We shall, however, sketch in the ideas which come up in the proof that a tangent exists at any point M of a circle. First choose a point P on one side of M and draw the secant MP. Then let P approach M as a limit. It may be shown that as P approaches M, the secant MP turns always in one direction about the point P. This comes of the fact that the circle is a convex curve. But MP does not turn as far as the line formed by producing the radius through P. Hence, analogously to Theorem 4, there exists a limiting direction of the secant MP. We have therefore established the existence of a tangential direction on one side of the point M. By taking the point P on the other side we may likewise establish the existence of a tangential direction on that side. These two directions may be shown to be opposite directions along the same line through M; and the proof of the existence of a tangent is then complete. With these suggestions we shall leave the problem to the reader.

ON COMPLETE SYMMETRIC FUNCTIONS.*

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PART I. INTRODUCTION.

1. Definitions, Notation, and Object of the Paper.

Let $\phi(\alpha_1, \alpha_2, \ldots\ldots, \alpha_n)$ be any rational function of the α's. Let $s=\begin{pmatrix}1\,2\,3 \ldots\ldots n\\ i_1 i_2 i_3 \ldots\ldots i_n\end{pmatrix}$, $i_1, i_2, \ldots\ldots, i_n$ being some permutation of $1, 2, 3, \ldots\ldots, n$; then s is an operator which converts the index r into i_r; also s applied to ϕ, converts it into ϕ_i, a function in which the α's have been permuted. This is expressed by writing

$$(1) \qquad s\phi=\phi_i.$$

Of such operators s there are $n!$ Applying each one to ϕ, we get $\phi_1, \phi_2, \ldots\ldots, \phi_{n!}$. Let

$$(2) \qquad \Phi=\phi_1+\phi_2+\ldots\ldots+\phi_{n!}.$$

Then Φ is a symmetric function of the α's. In particular cases it may happen that $\phi_1, \phi_2, \ldots\ldots, \phi_{n!}$, are not all different, but it can be proved that each dis-

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